'No Overlap No Gap' and the attack of a linear target by n different weapons

I David[1] and M Kress[2]*

[1]Ben-Gurion University of the Negev, Beer-Sheva, Israel; [2]Naval Postgraduate School, Monterey, CA, USA

David and Alalouf have specified the optimal aiming-points for Pattern Firing on a linear target by n identical munitions that are subject to systematic error only. Under the same distributional assumption on the error, namely, that its pdf is unimodal and symmetric around zero, we generalize these results to the case where the munitions (weapons) vary in lethality. Combat situations to which this generalization applies are also discussed.

Introduction

Pattern Firing occurs when a system fires a pre-assigned number of rounds to engage a given target more or less simultaneously, and it is capable of selecting individual aiming-points for these rounds (allowing 'offsets'). Many military OR problems concern Pattern Firing. Classical examples have been documented at length,[1] and they remain to be important in modern air-to-ground operations, naval operations, tank gunnery, missiles and canister artillery, to name only a few immediate application areas.

We consider the situation where round-to-round (ballistic) errors are negligible compared to the aiming (systematic) errors. This assumption may be valid in many firing situations such as aerial bombardment, which may be subject to relatively small dispersion but to relatively significant aiming error. David and Alalouf[2] noted that when the target and the distribution of the hit displacements are one-dimensional ('linear target'), and when there is no round-to-round error, the optimal aiming points that maximize the probability to hit the target by *at least* one of several identical rounds are easily calculable. Specifically, envision a linear target of width $2a$ (eg, a pipeline with cookie-cutter damage function of distance a) and a lateral systematic error for all n aiming points that is symmetrical, with, say, a standard deviation σ. If the origin is at the target midpoint (eg, the pipeline itself), then the optimal lateral aiming points c_i for rounds $i = 1, \ldots, n$ are independent of σ, and they are given by:

(a) If n is even, $n = 2k$: $c_1 = -(2k-1)a$, $c_2 = -(2k-3)a$, ..., $c_k = -a$, $c_{k+1} = a$, $c_{k+2} = 3a$, ..., $c_n = (2k-1)a$,

Correspondence: M Kress, Operations Research Department, Naval Postgraduate School, Monterey, CA 93943, USA.
E-mail: mkress@nps.edu

(b) If n is odd, $n = 2k+1$: $c_1 = -2ka$, $c_2 = -(2k-2)a$,..., $c_k = -2a$, $c_{k+1} = 0$, $c_{k+2} = 2a$, ..., $c_n = 2ka$.

This holds true for any unimodal error distribution, symmetric around zero (eg, a Gaussian unbiased error). Note that the aiming points are symmetrical around the target centre and are equally spaced along the length of the linear target (in the above application, along the *width* of the pipeline).

The rigorous proof in David and Alalouf[2] progresses gradually along three lemmas, and it can be visualized informally as follows: think of the problem as if there were a point target at the origin of an $x-y$ plane, and each delivery round is tantamount to sliding a stick, along the y-axis, in order to cover the origin. Each stick is of length $2a$. Firstly, there should be no holes ('gaps') between the sticks. This is shown by way of contradiction, using the two-sided drop-off of probability at the tails of the error-distribution. Secondly, once it is known that 'one big stick' is to be thrown, the same argument as above yields that the centre of the stick should be aimed at the point target. Thirdly and finally, it is now obvious that the stick has to be stretched as long as possible with overlapping among the individual sticks. This 'No Overlap No Gap' principle, together with symmetry, unequivocally give (a) and (b) above.

A generalization

The stick metaphor gives rise to the idea that the original sticks that belong to the n rounds need not be of the same size. They do not even have to evolve from symmetric vulnerability relative to the middle of the target (a target may be more vulnerable at one side than at the other). Specifically, we assume from now on that round i,

Report Documentation Page

1. REPORT DATE **22 DEC 2004**	2. REPORT TYPE	3. DATES COVERED **00-00-2004 to 00-00-2004**	
4. TITLE AND SUBTITLE **'No Overlap no Gap' and the attack of a linear target by n different weapons**		5a. CONTRACT NUMBER	
		5b. GRANT NUMBER	
		5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)		5d. PROJECT NUMBER	
		5e. TASK NUMBER	
		5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Naval Postgraduate School,Operations Research Department,Monterey,CA,93943**		8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)	
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**			
13. SUPPLEMENTARY NOTES			
14. ABSTRACT			
15. SUBJECT TERMS			

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT **Same as Report (SAR)**	18. NUMBER OF PAGES **4**	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**			

$i = 1, \ldots, n$, kills the linear target if it hits at most S_i meters before the target centre ('short'), or at most L_i meters beyond the target center ('long'). The difference among the individual S_i's and L_i's reflects the difference among the various munitions. It may also arise because of different target's *presented depths*: the 'linear' target in the plane may have a long second dimension, perpendicular to the direction at which the n rounds may be dropped. The attacker may choose to engage this target at various locations along its second dimension. Indeed, consider an aircraft carrying n bombs in order to hit a road or a pipeline. The only source of error is assumed to be the firing computer target location error, and it is common to all bombs. The bombs may incur different lethality against the road, or the bombs may still be identical, but the pilot, swooping down over the road more than once, elects to release them at different locations along that road. Airforce OR analysts may face such intriguing problems when target roads or other types of linear targets extend over water: if the bomb misses the physical boundaries of the line, that bomb is totally ineffective. However, if it hits the line, it is many-fold more effective than a bomb, which hits the line on the ground. The ensuing optimization problem of this case is, however, beyond the scope of the present prototypical model.

As before, let the aiming point of round i be c_i, relative to the target centre (eg, the median of a road). The new result is as follows.

Proposition *Suppose n rounds are fired subject to a systematic error only, with a symmetric, unimodal error-distribution. Then the optimal aiming points in the different-weapon case are given by:*

$$c_l = \sum_{i=0}^{l-1} (S_i + L_i)/2 - \sum_{i=l}^{n} (S_i + L_i)/2 + S_l \qquad (1)$$

independently of the error standard deviation σ, for $l = 1, \ldots, n$. S_0 and L_0 are artificially set to zero.

Proof We choose an orientation as shown in Figure 1: a 'long' hit is to the *left* of a point target on a one-dimensional axis. Thus, the target is killed by round i if the realization x of the aiming error X satisfies

$$-L_i \leqslant c_i + x \leqslant S_i$$

for all i, $i = 1, \ldots, n$, or, alternatively,

$$-L_i - c_i \leqslant x \leqslant S_i - c_i$$

(Recall that the aiming error is common to all rounds and therefore x need not be indexed). The interval $[-L_i - c_i, S_i - c_i]$ is called the Permissible Error Range (*PER*) of round i, and is denoted by PER_i. Any aiming error within PER_i results in round i being effective. Its length, $S_i + L_i$, may be
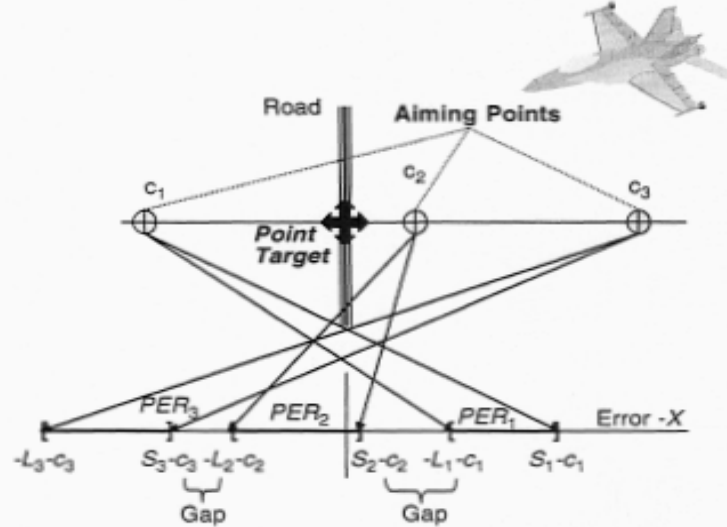

Figure 1 Aiming-points and their respective *PER*s.

thought of as the length of a 'stick' thrown in round i. Figure 1 visualizes this concept as well.

We establish the 'No Overlap No Gap' and Symmetry in the different-weapon case, and treat the '*No gap*' part first, that is, for all i, $i = 1, \ldots, n-1$, $c_{i+1} - c_i \leqslant L_i + S_{i+1}$. See Figure 1. If not, there is some j such that $c_{j+1} - c_j = L_j + S_{j+1} + h$, for $h > 0$. Assume that $c_j + L_j < 0$. (A *right* case: it is 'right' because PER_j lies to the right of zero).

1. Suppose $j = 1$. $c_1 = c_2 - L_1 - S_2 - h$. The probability that round 1 hits the target is $F(-c_1 + S_1) - F(-c_1 - L_1) = F(-c_2 + L_1 + S_2 + S_1 + h) - F(-c_2 + S_2 + h)$. The arguments of F are positive, since $-c_2 + S_2 + h = -c_1 - L_1 > 0$. Thus, because the derivative of F decreases for positive values, this probability is maximized with $h = 0$. (The differentiability of F may be relaxed, but it helps in presentation.) By replacing c_1 with $c_1^* = c_2 - L_1 - S_2$, we increase the total objective by the resulting positive difference for the first round.

2. Suppose $j > 1$. Distinguish between the case where the pair of aiming-points j and $(j-1)$ also makes a gap, and the case where it does not. If not, then $c_j - c_{j-1} < L_{j-1} + S_j$. Define $h^* = \min\{h, L_{j-1} + S_j - (c_j - c_{j-1})\}$. Change the jth aiming-point to $c_j^* = c_j + h^*$. (No change is made in the other aiming-points.) It follows that this increases the overall objective *exactly* by $P(-c_2 - L_2 - h^* \leqslant X \leqslant -c_2 - L_2)$. If, however, there is an additional gap there, that is, $c_j - c_{j-1} < L_{j-1} + S_j$ (see Figure 1), the condition of the jth round is the same as that of the first round in the case $j = 1$ above, and, by filling the gap in letting $h = 0$, we increase the objective probability exactly by the resulting probability-increase for the jth round.

If $c_j + L_j > 0$ (a *left* case), then necessarily $c_{j+1} - S_{j+1} > 0$, and a mirror analysis, distinguishing cases regarding round

$j+1$, may be carried out. In this time, pushing the gap down to $h=0$ relies on the *increase* of the density of F for *negative* values.

Now, with 'No gap' established, the resulting expression for the objective is $P(-c_n-L_n \leqslant X \leqslant -c_1+S_1)$. It is easy to see, using the same properties of F again, that this expression is maximized by letting $c_n+L_n=-(c_1+S_1)$. This is *symmetry*. The length of the total PER, *TPER*, is $2(c_n+L_n)$.

We conclude with showing the '*No overlap*' part. Indeed, by contradiction again, if there is a round j, $1<j<n-1$, such that $c_{j+1}-c_j+h=L_j+S_{j+1}$, for $h>0$, we redefine the aiming-points such that $c_i^*=c_i-h/2$ for $i=1,\dots,j-1$, and $c_i^*=c_i+h/2$ for $i=j,\dots,n$. Symmetry and 'No gap' are retained, and *TPER* is increased by h, increasing the total kill probability as well. Thus, the original set of aiming-points cannot be optimal.

(It is not difficult to extend the overall argument by proving that if P is a probability measure on the real line with a unimodal, symmetric density, and max $P(A)$ is sought over all Borel sets that have Lebesgue measure $2a>0$, then the solution is obtained taking A to be the interval $[-a, a]$. The extension is hardly of interest to air-bombers, though).

Letting now $1, 2, 3, \dots, n$ be the order of round delivery, we invoke the 'No Overlap No gap' and the symmetry principles, to get that the optimal c_i's must satisfy the following set of independent linear equations:

$$-L_1-c_1 = S_2-c_2 \qquad (\textit{No Overlap No Gap}) \qquad (2a)$$
$$-L_2-c_2 = S_3-c_3 \qquad ('') \qquad (2b)$$

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$$-L_{n-1}-c_{n-1} = S_n-c_n \qquad ('')$$
$$-L_n-c_n = -S_1+c_1 \qquad (\textit{symmetry}) \qquad (2n)$$

The result (1) may readily be seen to satisfy the set of independent set of linear equations (2a)–(2n). □

Examples and comments

n! different optimal aiming points. Since the round delivery sequence is arbitrary, and since all aiming-points must make a symmetric 'stick', it follows that the maximum kill probability may be attained by each one of the $n!$ possible orders of bomb releases. Each order induces, however, a different pattern that is optimal, as implied by the respective set of linear equations (2a)–(2n) above.

There are exactly $n!$ optimal solutions—so that the order of releasing bombs does not affect optimality and therefore it may be determined by operational considerations.

The kill probability. The total *PER* length, *TPER*, is obviously $\sum_{i=1}^{n}(S_i+L_i)$. If F is the cdf of X, then, by symmetry, the maximal kill probability is $2F(TPER/2)-1$.

Example: $n=2$, Gaussian distribution

In this case a rigorous proof of the main result is available straightforwardly, along traditional lines. Indeed,

$$P(X \in PER_1) + P(X \in PER_2) - P(X \in PER_1 \cap PER_2) \qquad (3)$$

needs to be maximized, over c_1 and c_2. We assume without loss of generality that $c_1 \leqslant c_2$.

Expression (3) translates to

$$\begin{aligned} f(c_1,c_2) = {} & \Phi(-c_1+S_1) - \Phi(-c_1-L_1) \\ & + \Phi(-c_2+S_2) - \Phi(-c_2-L_2) \\ & - \Phi(\min\{-c_1+S_1, -c_2+S_2\}) \\ & + \Phi(\max\{-c_1-L_1, -c_2-L_2\}) \end{aligned} \qquad (4)$$

constrained such that the min should be no smaller than the max. ($\sigma=1$ has been assumed, without loss of generality. Φ is the standard normal cdf.)

If $c_1+L_1< c_2-S_2$, the two PERs do not intersect, and f takes only the first two summands in (4). If we select c_2 such that $-c_2+S_2<0$, $\Phi(-c_1+S_1)-\Phi(-c_1-L_1)$ is increasing, and the expression is maximized by the rightmost c_1, giving

$$c_1 + L_1 = c_2 - S_2 \qquad (5)$$

In a similar way, if we start with c_2 such that $-c_2+S_2>0$, and any c_1 such that $c_1+L_1< c_2-S_2$, there is a possible redefinition of c_1 and c_2, obeying (5), which yields a better objective. Substituting (5) in the first line of (4), and maximizing the unconstrained problem in c_1, then applying (5) again, we readily get $c_1=(S_1-S_2-L_1-L_2)/2$, and $c_2=(S_1+S_2+L_1-L_2)/2$, complying with (1).

If $c_1+L_1 \geqslant c_2-S_2$, (intersecting PERs, one 'stick'), we shall assume that the min in (4) takes its value at $-c_2+S_2$ and the max at $-c_1-L_1$. (It is argued again that an equivalent such stick may be arranged in all other cases, by a proper choice of c_1 and c_2.) We further take for granted that the stick-aiming covers the point-target, that is, $c_2+L_2>0$ and $c_1-S_1<0$. Equation (4) reduces to

$$f(c_1,c_2) = \Phi(-c_1+S_1) - F(-c_2-L_2)$$

Using the identity $\phi'(x)=-x\phi(x)$ for the standard normal density, the sign of the determinant of the *Hessian* matrix of f comes out easily to be that of $-(c_1-S_1)\cdot(c_2+L_2)$, which is positive. The second derivative of f with regard to c_1 is positive too, so f is convex. It is constrained to the convex (trapezoidal) area defined by $c_2 \geqslant c_1$, $c_2 \leqslant c_1+L_1+S_2$, $c_1 \leqslant S_1$, and $c_2 \geqslant L_2$, and thus it takes its maximum on the boundaries of this area. It follows that (5) applies again, and the rest is as before. Evidently, carrying on this line of argument to higher n's is highly tedious.

A numerical example

Suppose *three* missiles are launched at an elongated vertical antenna (or a tall and thin command building). The missiles are aimed at three level points, where the antenna shape or vulnerability differ, so that $S = 30$ and $L = 60$ for missile 1, $S = 20$ and $L = 40$ for missile 2, and $S = 10$ and $L = 70$ for missile 3. The missiles share a common (horizontal) aiming Gaussian error (there are no vertical errors). The following table lists three of the six possible optimal firing sequences. The aiming displacements are horizontal, relative to the vertical centre-axis of the antenna.

Firing sequence	Optimal c_1	Optimal c_2	Optimal c_3	Total PER
$1 \rightarrow 2 \rightarrow 3$	-85	-5	45	$[-115, 115]$
$1 \rightarrow 3 \rightarrow 2$	-85	-15	75	$[-115, 115]$
$3 \rightarrow 2 \rightarrow 1$	-105	-15	55	$[-115, 115]$

If $\sigma < 50$, the odds of knocking down the antenna exceed 97%. If, on the contrary, $\sigma > 100$, these odds go down to less than 75% (see Figure 2).

Summary

Many military OR problems concern Pattern Firing where each round of fire is aimed at different point. Classical examples have been documented at length since the early 1950s, and they remain important in many modern combat situations such as air-to-ground operations, naval operations, tank gunnery, missiles and canister artillery. Optimizi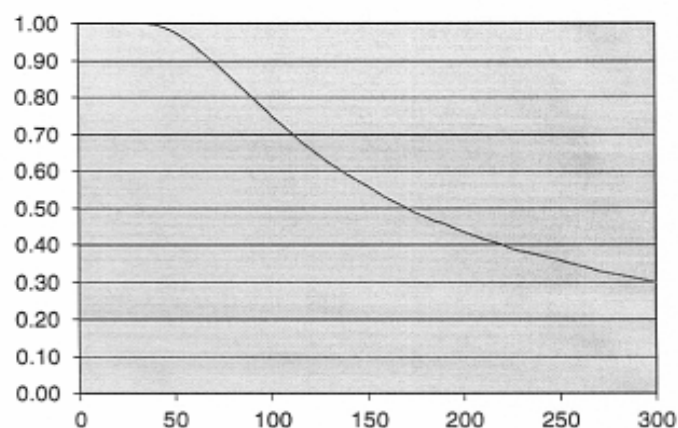ng the firing pattern is a typical problem in firing theory, which has been considered analytically only in a 'symmetric' setting where the munitions are identical. Modern weapon systems (eg, aircraft) typically carry several types of munitions and therefore optimizing multiple asymmetrical pattern firing has become an important problem. In this Note, we describe such a firing situation and obtain for the first time its optimal pattern.

Figure 2 Probability of knocking down the antenna as a function of σ.

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1 David I and Alalouf A (1999). On optimal aiming to hit a linear target. *Naval Res Logist* **46**: 323–333.
2 Morse PM and Kimball GE (1951). *Methods of Operations Research*. MIT Press: Cambridge, MA.